# Buckling, Postbuckling, and Nonlinear Vibrations of Imperfect Plates

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**Formulations and computational procedures are presented for studying the geometrically nonlinear behavior, including buckling, postbuckling, and nonlinear vibrations of perfect and imperfect, isotropic and laminated thin plates. The finite-element method is used. The element used is a 48 degree-of-freedom thin flat plate rectangular element capable of modeling arbitrary imperfections. The incremental and total stiffness matrices for large displacement behavior are derived based on the total Lagrangian approach in conjunction with the Hamilton's principle. The geometric imperfections are treated by considering additional terms in the strain-displacement relations. Numerical results are presented for a variety of examples including: 1) postbuckling analysis of an isotropic, imperfect flat plate; 2) postbuckling analysis of a thin, cross-ply laminated imperfect plate; 3) free vibrations of an angle-ply laminated plate; 4) free-vibrations of an imperfect isotropic plate under the action of axial loads; 5) nonlinear free-vibrations of isotropic perfect flat plates; 6) nonlinear vibrations of axially loaded, isotropic perfect flat plates; and 7) nonlinear vibrations of an imperfect laminated plate. The results obtained in this study are compared with existing solutions and a good agreement is seen.**

## Introduction

PLATE structures are used for a variety of functions in aerospace and other branches of engineering. For obvious advantages such as the critical strength-to-weight ratio, an increasing number of these structures are being made of laminated composites. During fabrication of both isotropic and laminated plate structures, it is possible that certain deviations between the actual and intended (desired) shapes may occur. These deviations or the so-called geometric imperfections, are generally quite localized and may significantly alter the structural behavior of these plates.

A review of the various developments in geometrically nonlinear behavior of thin plates has been given by Chia.[1] Yamaki et al.[2,3] studied the effect of geometric imperfections and initial edge inplane displacements on the postbuckling, small-amplitude free vibrations and large-amplitude vibrations of rectangular plates. In addition to theoretically predicted responses, Yamaki et al. also observed experimentally a variety of nonsymmetric and nonperiodic responses in connection with the internal resonance, combination resonance and dynamic snap-through phenomena.

The effect of geometric imperfections on the linear vibrations of simply supported flat plates under in-plane uniaxial or biaxial compression was studied by Ilanko and Dickinson[4,5] and by Hui and Leissa.[6] It was determined that a significant increase in natural frequencies may occur in the presence of imperfections. The effect of geometric imperfections on the large-amplitude vibrations of the rectangular plates with hysteresis damping was studied by Hui.[7] Celep[8] studied the effect of shear and rotatory inertia on the nonlinear vibrations of the imperfect plates. Research on the nonlinear vibrations of simply supported and clamped circular plates in the presence of imperfections was conducted by Hui.[9] Studies on the nonlinear vibrations of almost simply supported cylindrical shells were conducted by Kovalchuk and Krasnopolskaya.[10] Investigations on linear vibrations of shallow spherical shells were conducted by Hui.[11] Both the free and forced vibrations of imperfect cylindrical shells under base excitation were studied by Watawala and Nash.[12] Rosen and Singer[13] used a linearized version of Koiter's nonlinear equations with inertia terms included to study the influence of imperfections on the vibrations of thin cylindrical shells. The linearization was performed assuming the amplitude of vibrations to be infinitesimal so as to greatly simplify the analysis while retaining the imperfection terms. Also, Singer and Prucz[14] studied the influence of initial geometric imperfections on the vibrations of axially compressed stiffened cylindrical shells. Hui and Leissa[15] studied the effects of unidirectional geometric imperfections on vibrations of pressurized shallow spherical shells.

All of the forementioned studies were pertinent to the isotropic materials. Numerous studies have been conducted on the linear and nonlinear analyses of perfect and imperfect laminated plates. The analysis of laminated plates is complicated by the fact that even for infinitesimal amplitudes, there exists a coupling between bending and stretching behaviors (see for example, Refs. 16 and 17). Large amplitude vibrations of perfect laminated plates were studied in Refs. 18-25. Recently, Chia[26] studied the large amplitude vibration of unsymmetric angle-ply rectangular plates on elastic foundation having the varying rotational edge constraints.

The effect of geometric imperfections on the post-buckling behavior of imperfect laminated plates was studied by Hui.[27] Also examined were the effects of these imperfections on the linear vibrations of the biaxially compressed, antisymmetric angle ply rectangular plates, and also on the nonlinear vibrations of antisymmetrically laminated angle- and cross-ply simply supported and clamped rectangular thin plates.[28,29] It

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was observed that the presence of imperfection amplitudes, of the order of only one-half the total laminated plate thickness, may significantly alter the vibration frequencies and change the large amplitude vibration behavior from the well known hard-spring to soft-spring behavior.

In all the forementioned analytical studies, the imperfection shapes were chosen to be the same as the mode shapes, and the results were obtained for simply supported or clamped rectangular plates. In actual practice, the imperfection shapes are arbitrary and may be quite localized. Furthermore, the boundary conditions may also be arbitrary. For these cases, the finite-element method is most suitable.

A review of various finite-element developments on geometrically nonlinear analysis of plates and shells was given by the present authors,[30,31] Bushnell,[32] and by Reddy and Chao.[33] A 54 degree-of-freedom triangular flat plate element was developed by Yang and Han[34] to study the geometrically nonlinear behavior of isotropic flat plates. An earlier finite-element work on geometrically nonlinear behavior of laminated shells was by Noor and Mathers[35] using 8-noded and 12-noded shear-flexible quadrilateral shell elements. A hybrid formulation of a 6-noded triangular element and an 8-noded quadrilateral element including the effect of shear deformation was given by Noor and Hartley.[36] Chang and Swamiphakadi[37] presented the formulations of a degenerated three-dimensional isoparametric element. Reddy[38] and Reddy and Chandrashekhara[39] presented results for large deflection analysis including the effects of transverse shear deformation.

The finite-element method has extensively been used for studying nonlinear vibrations. A review of the various efforts can be found in Ref. 34. The method has been employed by Mei,[40] Rao, Raju, and Raju,[41,42] and Reddy.[43] Generalized incremental Hamilton's principle and element formulation was used by Lau et al.[44,45] to study the nonlinear vibrations of thin elastic plates using a triangular plate element with 15 stretching and bending nodal displacements based on an incremental modified discrete Kirchhoff theory (DKT). The application of the finite-element method to the analysis of imperfect isotropic and laminated anisotropic plates appears to be lacking.

The present paper is concerned with the buckling, postbuckling, and nonlinear vibrations of the imperfect plates. A 48 DOF rectangular plate element[30] is used to perform the analyses. The formulation is general, with respect to the geometry of the imperfections, boundary conditions, and number, orientation, and stacking sequence of the lamina. The plate is assumed to be very thin so that the classical laminate theory is applicable. The geometric imperfections are treated by considering additional terms in the strain-displacement relations. A variety of examples with both perfect and imperfect configurations are solved. The present results are compared with the existing alternative solutions to check the accuracy of the present developments.

## Imperfect Plate Element

The element used is a rectangular plate element. The thin shell version of this element, which is capable of modeling arbitrary geometric imperfections, was previously presented.[30] The element was later extended to study the large displacement behavior of laminated anisotropic shells.[31] The geometric imperfections considered are the small deviations of the actual fabricated plate from the intended or the desired shape. These imperfections are in the structure geometry and not in any given laminate. A summary of the salient features of the element formulation is given in the following sections.

### Displacement Functions

The element is rectangular in shape and has four nodes, one at each corner. Each node possesses 12 DOF's: $u^1$, $\partial u^1/\partial\xi$, $\partial u^1/\partial\eta$; $\partial^2 u^1/\partial\xi\partial\eta$ and similar quantities for $u^2$ and $u^3$, where $u^1$, $u^2$, and $u^3$ are the displacement components in the Cartesian directions, respectively, and $\xi$ and $\eta$ are the local coordinates on the middle surface of the shell. The displacement functions for $u^1$, $u^2$, and $u^3$ are expressed in terms of bicubic Hermitian polynomials in $\xi$ and $\eta$.

### Geometric Imperfections

The Cartesian components of the imperfections $v^1$, $v^2$, and $v^3$ of the middle surface of the plate finite-element are described by polynomial functions of the local coordinates $\xi$ and $\eta$:

$$v^i(\xi,\eta) = \sum_{j=1}^{N} \bar{c}^i_j \xi^{m_j} \eta^{n_j} \qquad i=1,2,3 \tag{1}$$

where the constants $m_j$ and $n_j$ define the indices of $\xi$ and $\eta$, respectively, for the $j$th term. The constants $\bar{c}^i_j$ are determined based on the imperfections $v^i$ $(i=1,2,3)$ at $N$ selected points on the middle surface of the plate element. It is noted that the above definition allows the geometry of the imperfections to be fairly general and not subjected to restrictions such as the imperfections being sinusoidal.

### Strain-Displacement Relations

The strain-displacement relations are modified to include the effects of imperfections. The tangential strain components can be written as[30]

$$\epsilon_{\alpha\beta} = ½\,[f^i{}_{,\alpha}u^i{}_{,\beta} + f^i{}_{,\beta}u^i{}_{,\alpha} + u^i{}_{,\alpha}u^i{}_{,\beta} + v^i{}_{,\alpha}u^i{}_{,\beta} + v^i{}_{,\beta}u^i{}_{,\alpha}] \tag{2}$$

where $\alpha$, $\beta$ varies from 1 to 2 and $i$ from 1 to 3 ($u^i$ and $v^i$ have been defined previously). Also, $f^1=x^1$, $f^2=x^2$ and $f^3=x^3$ coordinates. The effect of imperfections on the curvature-displacement relations is ignored. Thus, the curvature-displacement relations are the same as those for flat plates.

### Laminate Constitutive Relations

The laminated anisotropic construction of the plate is assumed to be made up of $n$ layers. Each lamina is assumed to be orthotropic with its principal material axes at an angle $\theta$ with the $\eta$-$\xi$ axes. The stress-strain relation for each layer is referred to the $\eta$-$\xi$ system using a coordinate transformation. The stress and moment resultants are then related to the midsurface strains $\{\epsilon\}$ and change of curvature $\kappa$, as given in Ref. 46.

### Element Formulation

The equations of motion can be derived using the Hamilton's principle, namely

$$\delta \int_{t_1}^{t_2} (T-U-V)\,\mathrm{d}t = 0 \tag{3}$$

where $T$ is the kinetic energy, $U$ the strain energy and $V$ the potential energy of the applied loads. Equation (3) gives the Lagrange equations of motion which can be written as

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad (i=1,2...48) \tag{4}$$

where $L=T-U-V$ is called the Lagrangian, and $q_i$ $(i=1,2,...48)$ are the degrees-of-freedom of the element. Substituting the displacement functions in Eq. 4, the element equations of motion can be obtained as

$$[[k] + ½[n_1] + ⅓[n_2]]\{q_e\} + [m]\{\ddot{q}_e\} = \{p_e\} \tag{5}$$

where $k$ is the linear stiffness matrix, and $[n_1]$ and $[n_2]$ the first- and second-order stiffness matrices, respectively. A linear function of the element displacements is $[n_1]$ and a quadratic function of the element displacements is $[n_2]$; $\{p_e\}$ is the total load vector.

It is noted that the above matrices can be written in the incremental form as

$$[[k]+[n_1]+[n_2]]\{\Delta q_e\}+[m]\{\Delta\ddot{q}_e\}=\{\Delta p_e\} \qquad (6)$$

The above equation can also be written as

$$[k_T]\{\Delta q_e\}+[m]\{\Delta\ddot{q}_e\}=\{\Delta p_e\} \qquad (7)$$

The matrix $[k_T]$ is called the tangent stiffness matrix and is written as

$$[k_T]=1\iint_A [B_0+B_1+B_L(q_e)]^T\left[\begin{array}{c|c} A & B \\ \hline B & D\end{array}\right][B_0+B_i+B_L(q_e)]\,\mathrm{d}A+\iint_A [G]^T[H][G]\,\mathrm{d}A \qquad (8)$$

The matrices $[B_0]$, $[B_i]$, and $[B_L(q_e)]$ relate the incremental strains $\{\mathrm{d}\epsilon\}$ at $\{q_e\}$ to the incremental displacements $\{\Delta q_e\}$, as

$$\{\mathrm{d}\epsilon\}=[B_0+B_i+B_L(q_e)]\{\Delta q_e\} \qquad (9)$$

Note that $[B_0]$ is the same as that for the linear case. The strain-displacement matrix due to the imperfections is $[B_i]$ and the contribution of the nonlinear terms is $[B_L]$. Matrices $[G]$ and $[H]$ are given by Kapania and Yang.[30]

## Solution Procedure

### Postbuckling and Large Displacement Analysis

On assembling the finite-element and applying the kinematic boundary conditions, the statically nonlinear equilibrium equations may be written as

$$\{P\}=[[K]+[N_1]/2+[N_2]/3]\{Q\} \qquad (10)$$

Equation (10) can be solved by using the Newton-Raphson method which may be expressed symbolically as

$$[K_T]\{\Delta Q\}_{i+1}=\{\Delta P\}_i \qquad (11)$$

$$\{\Delta P\}_i=\{P\}-\{P_i\} \qquad (12)$$

where $\{P_i\}$ is the total internal load, and for a given element (before assemblage) as

$$\{P_i^e\}=\iint_{A_i}[B_0+B_i+B_L(q_e)]_i^T\left\{\begin{array}{c} N_{\alpha\beta}(q_e) \\ M_{\alpha\beta}(q_e)\end{array}\right\}_i \mathrm{d}A \qquad (13)$$

By solving Eq. (11), the improved displacement vector is given as

$$\{Q\}_{i+1}=\{Q\}_i+\{\Delta Q\}_{i+1} \qquad (14)$$

where subscript $i$ denotes the $i$th iteration cycle. The tangent stiffness matrix $[K_T]$ may or may not be updated in every iteration cycle.

The following convergence criteria was used to terminate the iterations at every incremental step:

$$\left[\left(\sum_{i=1}^{N}\Delta P_i^2\right)\Big/\left(\sum_{i=1}^{N}P_i^2\right)\right]^{1/2}\leqq 0.1\%$$

$$\left[\left(\sum_{i=1}^{N}\Delta Q_i^2\right)\Big/\left(\sum_{i=1}^{N}Q_i^2\right)\right]^{1/2}\leqq 0.1\% \qquad (15)$$

where the subscript $i$ is the degree-of-freedom number and $N$ is the total number of degrees-of-freedom of the finite-element modeling.

### Nonlinear Vibrations

The equations of motion for nonlinear vibrations may be written as

$$\left[\begin{array}{c|c} K_{uu} & K_{uw} \\ \hline K_{wu} & K_{ww}\end{array}\right]\left\{\begin{array}{c} Q_u \\ \hline Q_w\end{array}\right\}+\frac{1}{2}\left[\begin{array}{c|c} 0 & N_{uw_1} \\ \hline N_{wu_1} & N_{ww_1}\end{array}\right]\left\{\begin{array}{c} Q_u \\ \hline Q_w\end{array}\right\}+\frac{1}{3}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & N_{ww_2}\end{array}\right]\left\{\begin{array}{c} Q_u \\ \hline Q_w\end{array}\right\}+\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{ww}\end{array}\right]\left\{\begin{array}{c} \ddot{Q}_u \\ \hline \ddot{Q}_w\end{array}\right\}=\left\{\begin{array}{c} 0 \\ \hline 0\end{array}\right\} \qquad (16)$$

where the subscript $u$ refers to the in-plane displacements $u^1$ and $u^2$, and the subscript $w$ refers to the normal displacements $u^3$. The in-plane inertia is being neglected in this study.

For the case of isotropic, perfect plates $[K_{uw}]=[K_{wu}]=0$. In this study, the $[N_1]$ matrix was written in such a way that $[N_{ww_1}]=0$ if the plate is perfect and isotropic: this renders the $[N_1]$ matrix unsymmetrical (for an explanation of this, the reader is referred to Ref. 47). For the case of laminated or imperfect plates, the submatrix $[N_{ww_1}]$ is nonzero and is a function of $\{Q_w\}$.

For the case of perfect, isotropic plates, the same procedure as used by Yang and Han[34] was used in this study. However, for the case of isotropic and laminated imperfect, or for the perfect, anisotropic plates, the procedure used by Yang and Han[34] can not be applied, since the motion can no longer be written in terms of $a_1 \cos\omega t$ alone. Additional terms like $a_0$ and $a_2 \cos 2\omega t$ may play a significant role. A somewhat modified procedure, which uses the assumed-mode method will be used. The variation of the mode is assumed to be the same as the linear mode. This assumption is valid for moderately large amplitude vibrations. For very high amplitudes, the effect of other modes will need to be considered, that is, a multimode analysis will be needed. Let

$$\{Q_w\}=\{\bar{Q}_w\}\tau(t) \qquad (17)$$

where $\{\bar{Q}_w\}$ is the linear mode normalized so that the maximum displacement is unity, and $\tau(t)$ is a time function. For the sake of clarity, $\tau(t)$ will be written as $\tau$. Substituting Eq. (17) into Eq. (16), the in-plane displacement vector can be written as

$$\{Q_u\}=-[K_{uu}]^{-1}[K_{uw}]\{\bar{Q}_w\}\tau-½[K_{uu}^{-1}][N_{uw_1}]\{Q_w\}(\tau)^2 \qquad (18)$$

Substituting Eq. (18) in Eq. (16), the equation of motion for normal displacement can be written as

$$[K_{ww}-K_{wu}K_{uu}^{-1}K_{uw}]\{\bar{Q}_w\}\tau+[N_{ww_1}-½K_{wu}K_{uu}^{-1}N_{uw_1}-½N_{wu_1}K_{uu}^{-1}K_{uw}]\{\bar{Q}_w\}(\tau)^2+[⅓N_{ww_2}-¼N_{wu_1}K_{uu}^{-1}N_{uw_1}]\{Q_w\}(\tau)^3+[M_{ww}]\{\bar{Q}_w\}\ddot{\tau}=\{0\} \qquad (19)$$

For the case of isotropic perfect plates, the terms containing $(\tau^2)$ will be absent. The effect of $(\tau)^2$ terms is to change the nonlinearity from a hardening type [due to $(\tau)^3$ terms] to a softening type, depending upon the amplitude of imperfections (see, e.g., Refs. 48 and 28). Equations (18) and (19) can be solved by multiplying Eq. (19) by $\{\bar{Q}_w\}^T$ and solving the resulting nonlinear equation to obtain $\tau$. The function $\tau$ can be obtained using the method of Harmonic Balancing,[48] namely by assuming

$$\tau = A_0 + A_1 \cos\omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t + \cdots \tag{20}$$

where $A_i$ are the amplitudes of $\cos\omega_i t$ terms. Here $A_1$ is specified (the amplitude of vibrations) and $\omega$, $A_0$, $A_2$, and $A_3$ are determined.

## Numerical Examples

The forementioned formulations were used to study the buckling, postbuckling and nonlinear vibrations of perfect and imperfect isotropic and laminated plates. In all of the examples studied, a $5\times 5$ Gauss integration was used to evaluate the integrals over the area of the element. The calculations were performed at Virginia Polytechnic Institute and State University on a IBM 370 computer with a 3084 processor complex.

### Postbuckling Analysis of an Imperfect Isotropic Plate

The postbuckling analysis was performed for an imperfect isotropic square plate ($\nu = 1/3$) with all edges simply supported. The loaded edges were maintained straight with zero shear stress and the unloaded edges were kept stress-free and thus were free to wave in the plane of the plate. A $2\times 2$ finite-element mesh was used to idealize a quadrant of the plate. For the case of the perfect plate, the Euler buckling load and the corresponding mode shape were found first. This mode shape was then scaled up by an assumed deflection factor to form the nonlinear stiffness matrices. The results for center deflection vs inplane compression are compared with alternative results given by Coan[49] (see Fig. 1). A good agreement is observed. The effect of imperfections on the postbuckling analysis was then studied. Symmetric imperfections of the form

$$\mu = \mu_0 h \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \tag{21}$$

were assumed. The Newton-Raphson method was used to obtain the center deflection for $\mu_0 = 0.1$ and 0.25. For imperfect plates the central deflection is non-zero for loads less than the Euler buckling value. At higher compressive loads, the central deflections of the two imperfect plates tend to approach each other and the central deflection for the perfect plate.

### Postbuckling Analysis of Laminated Plates

Postbuckling analysis of a simply supported, graphite-epoxy cross-ply square plate was performed. Two cases, with two and four plies, respectively, were considered. In both cases, the loaded edge was considered to be free from shear stress. The unloaded edges were free to wave in the plane of the plate. The material properties for graphite epoxy were considered as $E_L/E_T = 40$, $G_{LT}/E_T = 0.5$, and $\nu_{LT} = 0.25$. The imperfections were considered to be of the same form as in the previous example [see Eq. (21)].

A $2\times 2$ finite-element mesh was used to idealize a quadrant of the plate. The response under axial load was obtained for $\mu_0 = 0.0$, 0.1, and 0.25. The results are presented in Fig. 2 for both plates. For perfect plates, the present results are in excellent agreement with those given by Chia.[1] For the case of cross-ply laminated plates, a very small axial load causes normal deflection of the plate, even for perfect plates.

### Linear Vibrations of an Angle-Ply Laminated Plate

After evaluating the present program for nonlinear static analysis, the present developments were evaluated for linear vibration analysis. A free-vibration analysis for a laminated angle-ply ($\pm\theta$) plate of graphite epoxy was performed. Two cases, with $N=2$ and $N=\infty$ were considered. Here $N$ is the number of layers. A $2\times 2$ mesh was used to model a quadrant of a simply supported square plate. Natural frequencies for the two cases for different values of fiber angle $\theta$ are given in Fig. 3. For comparison, the results given by Hui[28] are also presented. A good agreement is observed.

### Linear Vibrations of an Imperfect Isotropic Plate Under Axial Compression

The effect of axial load on the natural frequencies of an imperfect, isotropic simply supported square plate was studied. The imperfections considered were of the same form as given in Eq. (21). Three values of imperfect amplitudes were considered, namely, $\mu_0 = 0$, 0.25, and 0.5. For the case of imperfect plates, equilibrium state was first obtained. The linear vibrations about the equilibrium state were then obtained using the incremental dynamic equations given by Eq. (7).

A $2\times 2$ mesh was used to model a quadrant of the plate. Figure 4 shows the variation of $(\omega/\omega_0)^2$ with axial load. Here $\omega_0$ is the natural frequency at zero load. As expected, the variation is linear for $\mu_0 = 0$ and meets the $x$ axis as axial load becomes equal to the Euler buckling load. However, in the presence of imperfections, the natural frequency does not approach zero as the axial load is increased. The value of $\omega$

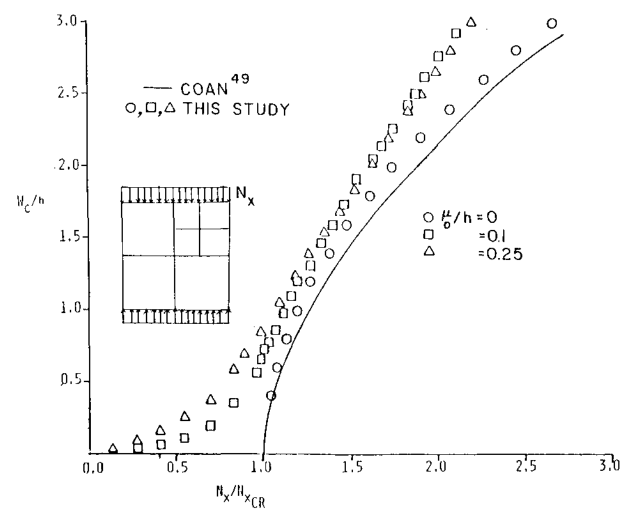


**Fig. 1 Postbuckling analysis of an isotropic imperfect simply supported square plate ($\nu = 0.333$).**

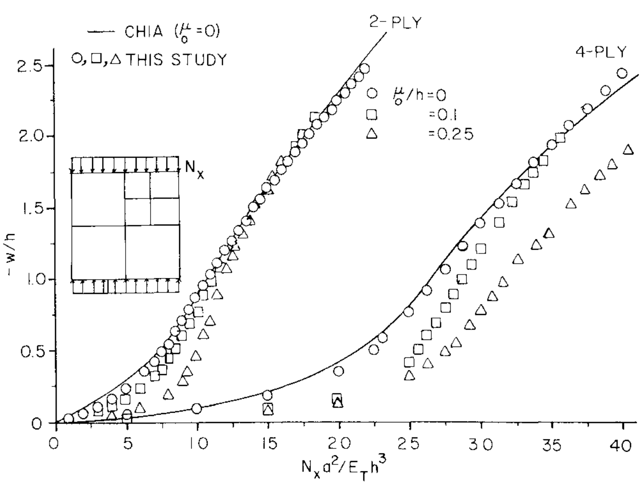


**Fig. 2 Postbuckling analysis of a two- and four-ply cross-ply laminated simply supported square plate.**

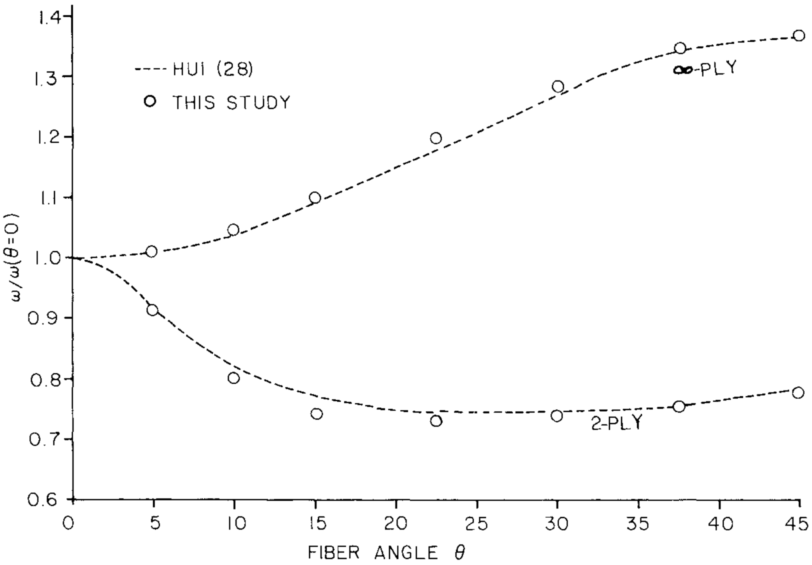


**Fig. 3 Normalized fundamental frequency vs fiber angle for a simply supported square two-ply angle-ply plate.**

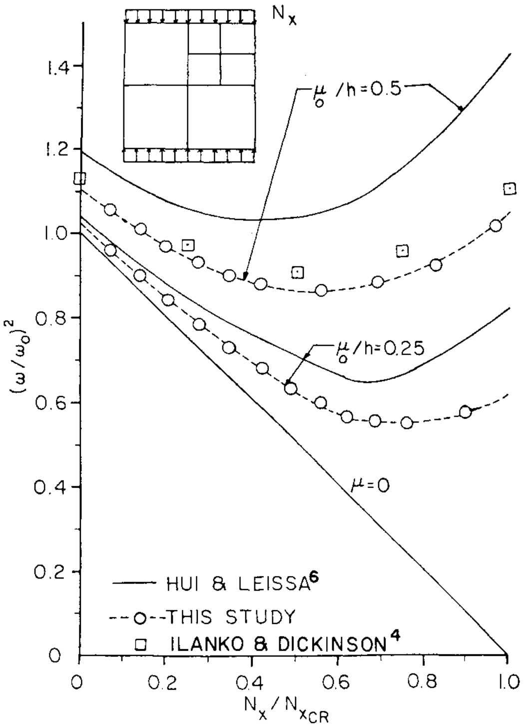


**Fig. 4 Variation of $(\omega/\omega_0)^2$ with in-plane load for different values of imperfection amplitude ($\nu = 0.3$).**

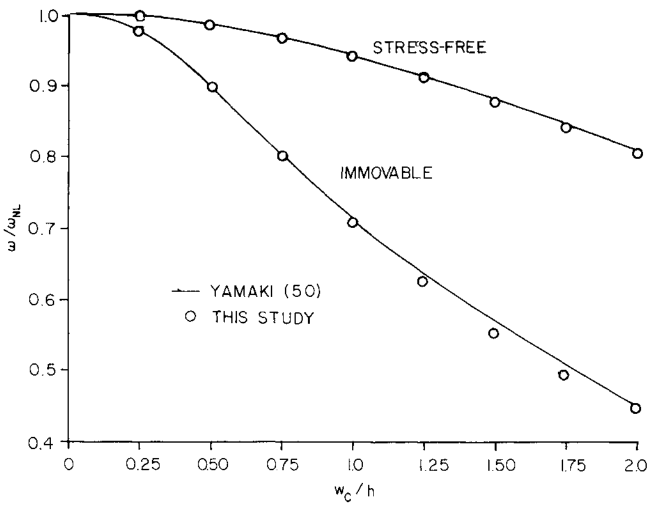


**Fig. 5 Variation of nonlinear/linear period for a simply supported square isotropic plate ($\nu = 0.3$).**

first decreases with the axial load and then increases. Similar trends were also observed by Ilanko and Dickinson[4,5] and by Hui and Leissa,[6] as shown in Fig. 4. The present results are in good agreement with the results obtained by Ilanko and Dickinson using the Rayleigh-Ritz method. It may be pointed out here, that Ilanko and Dickinson have also conducted experiments to study the effect of imperfections on the vibrations of axially loaded plates and have found a very good correlation between experimental and analytical results.[4,5] The present results, however, are quantitatively somewhat different from those obtained by Hui and Leissa. The difference is because the boundary conditions for the in-plane displacements are different in the two analyses, due to the mixed approach used in Ref. 6.

### Large-Amplitude Vibrations of Perfect Isotropic Plate

The element was then used to study the nonlinear vibrations of perfect isotropic plate. The analysis was performed for simply supported and clamped plates. Two in-plane boundary conditions were considered: 1) edges immovable and 2) edges stress free.

Large-amplitude vibration analyses were performed by Yamaki using a one-term Galerkin's method,[50] by Mei et al. using a 18 DOF triangular plate element,[51] and by Yang and Han using a sophisticated 54 DOF triangular plate element.[34] In Mei's analysis[51] no in-plane degrees-of-freedom were considered, and the edges for all example plates were fixed from in-plane displacements.

The results for the first mode period vs center amplitude, obtained by $2 \times 2$ mesh for a quadrant, and those by Yamaki[50] are shown in Fig. 5 for the case of a simply supported plate. (Results by Mei et al.[51] and Yang and Han[34] are not shown, as these were close to those of Yamaki.) It is observed that the present results are in excellent agreement with those obtained by Yamaki using one-term Galerkin's method.

Results for the clamped plate are shown in Fig. 6. For the case of immovable edges, the present results are in excellent agreement with those obtained by Yamaki[51] using the one-term Galerkin's method. For the case of stress-free edges, the present results for the nonlinear period are higher than the corresponding results by Yamaki with the largest discrepancy being 7%. The results obtained by Yang and Han[34] were also about 7% higher than those obtained by Yamaki.[50]

### Large-Amplitude Vibrations of Plates with Initial Stresses

The element was next employed to study the nonlinear vibrations of perfect isotropic plates under equal biaxial uniform edge compressive stress $\sigma_0$. The stress $\sigma_0$ produces two equal orthogonal uniform edge displacements $u_0$ and $v_0$. When $\sigma_0$ reaches its critical buckling value, the corresponding $u_0$ and $v_0$ are defined as $u_{cr}$. The analysis was performed for both simply supported and clamped plates. The edges were considered to be immovable for both the plates.

This problem was previously studied using a one term Galerkin's method[52] and by using a sophisticated 54 degree-of-freedom triangular plate finite-element.[34] The results for the effect of $\lambda = u_0/u_{cr}$ on the curves for nonlinear frequency vs center amplitude to thickness ratio, obtained by $2\times2$ mesh for a quadrant, and those by Eisley[52] are shown in Fig. 7 for the case of a simply supported plate. (The results by Yang and Han[34] are not being shown, as these were similar to those of Eisley.[52]) The present results are in excellent agreement with those obtained by Eisley.[52]

For the case of $u_0 = v_0 = 0$, both the present and Eisley's curves are identical to those obtained in Fig. 5 of this study and Yamaki's study, respectively, for the edge immovable case.

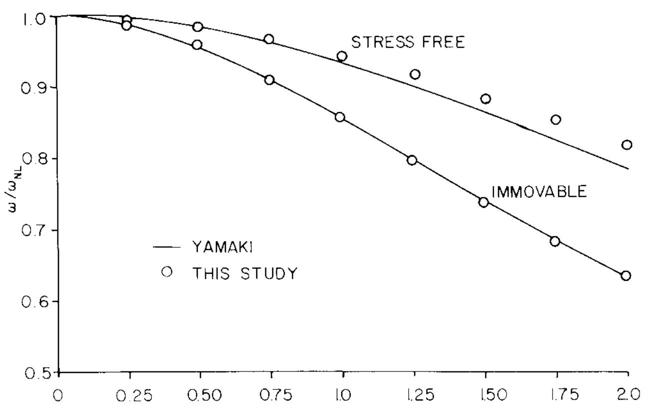


**Fig. 6 Variation of nonlinear/linear period for a clamped isotropic square plate ($\nu = 0.3$).**

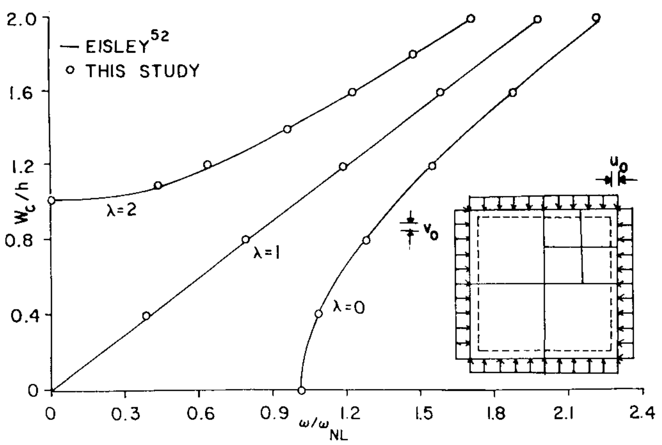


**Fig. 7 First mode large amplitude vibrations of a simply supported square plate with initial stresses ($\nu = 0.3$).**

For the case $u_0/u_{cr} = 1$, $\sigma_0$ equals the Euler buckling stress $\sigma_{cr}$ and the determinant of the sum of the linear stiffness matrix and the initial stress matrix vanishes. As a result, the nonlinear free-vibration eigenvalue equations are dominated by the mass matrix which contains $\omega^2$ and the incremental stiffness matrices which contain quadratic terms of amplitudes. Thus the amplitude-frequency curve appears to be linear.

For $u_0/u_{cr} = 2$, the present solution and that of Eisley are based on the assumption that the vibration of the plate is symmetric about the flat position. Another type of motion is possible in which the plate vibrates about a static buckled position on one side of the flat position. Such motion is not considered in the present paper.

The results for the clamped square plate are shown in Fig. 8. The trends for all the curves are quite similar to those shown in Fig. 7.

### Large-Amplitude Vibrations of Imperfect Laminated Plates

Finally, the present developments were applied to study the large-amplitude vibrations of imperfect laminated plates. The analysis was performed for a clamped two-layer-graphite epoxy cross-ply plate, and simply supported, orthotropic (2-ply, angle-ply plate with $\theta = \pm 0°$) plate. The edges were assumed to be immovable for both of these cases.

For the case of clamped, cross-ply plate, the imperfections were assumed to be given by

$$\mu = \mu_0 \frac{h}{4}\left[1 - \cos\frac{2\pi x}{a}\right]\left[1 - \cos\frac{2\pi y}{a}\right] \tag{22}$$

where $\mu_0$ is the amplitude of the imperfection at the center of the plate. For the case of a simply supported plate, the imperfections were assumed to be the same as these given by Eq. (21).

For both plates, the results were obtained using a $2\times2$ mesh to model a quadrant. Figure 9 shows the variation of the linear natural frequency of the clamped cross-ply plate with the imperfection amplitude. For comparison, the results obtained by Hui[29] are also shown. The two sets of results are in good agreement with each other. For higher imperfection amplitudes, the present study yielded the natural frequencies somewhat lower than those predicted by Hui.[29]

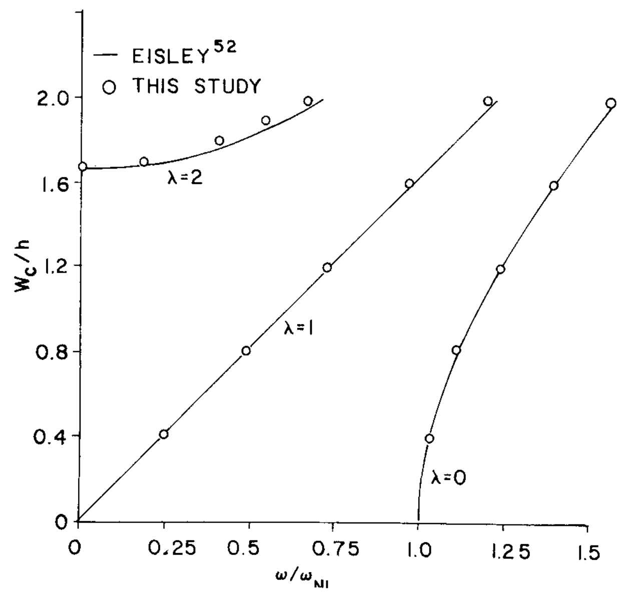


**Fig. 8 First mode large amplitude vibrations of a clamped square plate with initial stresses ($\nu = 0.3$).**

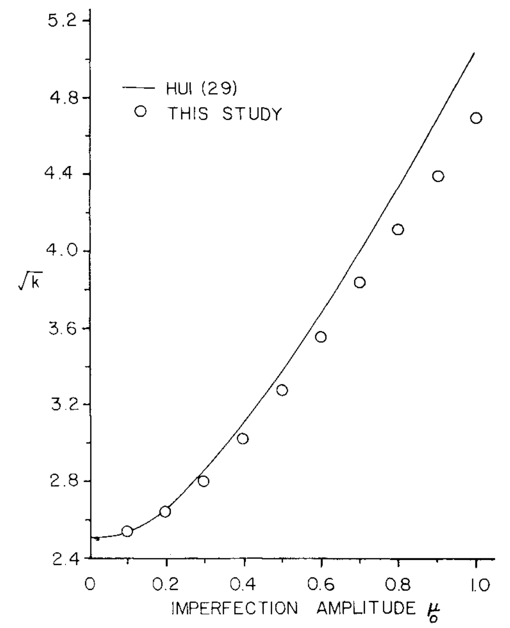


**Fig. 9 Linear frequency vs imperfection amplitude for clamped graphite-epoxy cross-ply square plate.**

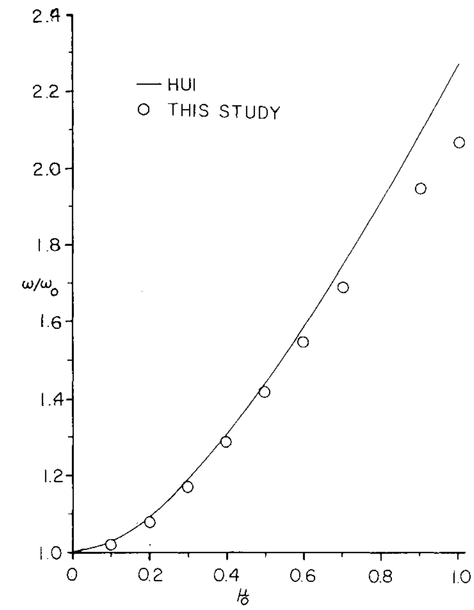


**Fig. 11 Linear frequency vs imperfection amplitude for simply supported graphite-epoxy two-ply angle-ply square plate.**

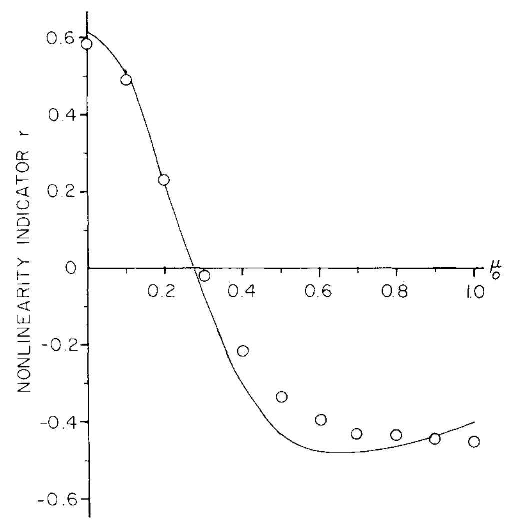


**Fig. 10 Nonlinearity indicator vs imperfection amplitude for clamped graphite-epoxy cross-ply square plate.**

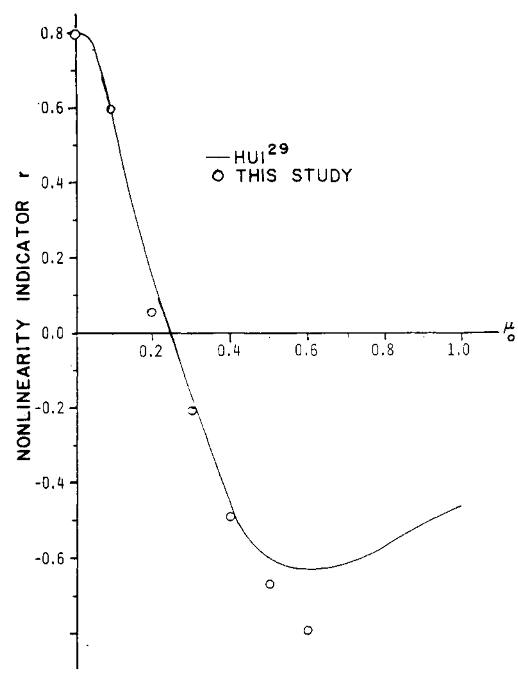


**Fig. 12 Nonlinearity indicator vs imperfection amplitude for simply supported graphite-epoxy two-ply angle-ply square plate.**

For the nonlinear vibrations, Hui[29] presented results in the form of a nonlinearity indicator $r$ defined as

$$\omega_{NL}/\omega = 1 + rA^2 \tag{23}$$

where $\omega_{NL}$ is the nonlinear frequency at amplitude $A$ and $\omega$ is the linear frequency. The above relation is valid only for moderately nonlinear problems. Figure 10 shows the variation of the nonlinearity indicator $r$ with imperfection amplitude. It is seen that as the imperfection amplitude increases, the value of $r$ changes from a positive value to a negative one. This indicates that the nonlinearity has changed from a hard-spring to a soft-spring. The results for $r$ from Ref. 29 are also shown in Fig. 10. Two sets of results are in good agreement with each other.

Figure 11 shows the variation of linear frequency with the imperfection amplitude for the case of a simply supported angle-ply plate. The results from Hui[29] are also shown. The two sets of results appear to be in good agreement, especially at low-imperfection amplitudes. For higher-imperfection amplitudes the present analysis predicts frequencies somewhat lower than those given by Hui.[29] The results for the nonlinearity indicator $r$ for the case of an angle-ply plate are shown in Fig. 12, along with the results obtained in Hui.[29] It is seen that the two sets of results are in good agreement with each other up to $\mu_0 = 0.5$. The present analysis predicts further softening. The results obtained for $\mu_0 > 0.6$ using this analysis are not shown.

It is believed that the difference in results obtained in the present work and those obtained by Hui[29] is due to the mixed approach used in Ref. 29. In that approach the in-plane boundary conditions can only be satisfied on the average, and it is especially difficult to satisfy immovable boundary conditions. On the other hand, in the displacement based finite-element analysis, the immovable edge conditions are satisfied exactly. But the finite-element solution cannot be that accurate for stress-free edges and also when edges are kept straight. For the analysis of unsymmetric and even for isotropic imperfect plates, the in-plane boundary conditions are known to play a significant role.

## Concluding Remarks

A review of the existing literature had shown the need for the development of a finite-element computer program that could perform the nonlinear vibrations of imperfect isotropic and laminated plates having arbitrary imperfections. A 48 degree-of-freedom thin rectangular plate element, the thin shell version of which was previously developed by the referenced authors for static analysis, was extended to study the large amplitude vibrations. The imperfections were represented by a variable-order polynomial so that any arbitrary imperfections could be included. The study of large-amplitude vibrations of the imperfect plates is made complex by the fact that the imperfections give rise to terms that are proportional to the square of the response amplitude in addition to the cubic terms. This implies that for certain values of imperfection amplitudes, the nonlinearity may change from a hardening type to a softening type.

The present developments were systematically evaluated for a series of examples related to the buckling and postbuckling of laminated plates, linear frequencies of imperfect plates under the effect of axial compression, nonlinear vibrations of perfect isotropic plates with and without initial stresses, and finally, nonlinear vibrations of imperfect laminated plates. The present results were found to be in good agreement with the available results.

The present developments can straight-forwardly be applied to shells for which the complete description of the imperfections is known. In fact, in one of our studies, the element was used to study the postbuckling behavior of a laminated cylinder having measured imperfections. The present formulations are being used to study the large-amplitude vibrations of imperfect shells. However, for shells (especially for shells of revolution) there is a strong interaction between different modes having the circumferential waves $n$, $2n$,...etc. This implies that for such shells, a multimode analysis will be needed. Present developments are being extended to include multimodes. Finally, the influence of transverse shear must be included for laminated plates and shells. The transverse stress effects are significant in the geometrically nonlinear range and in the dynamic analysis (see, for example, Refs 35, 36, 38, and 39).

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